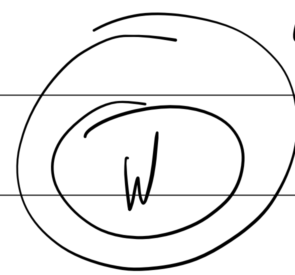
Due Fri

4.2 – Subspaces

Definition: A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 4.2.1 Subspace Test
If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.
a) If **u** and **v** are vectors in W, then **u** + **v** is in W. 1
b) If k is a scalar and **u** is a vector in W, then k**u** is in W. 6



V: obj, +, scalar mult, 10 axioms

Since any vector $\vec{w} \in W$ is also in V, axioms 2, 3, 7, 8, 9, 10 are satisfied. To show $W \subset V$ is itself a subspace we need to confirm 1, 4, 5, 6.

for all

But if 6 holds ~~for~~ W, and $\vec{w} \in W$, then $k\vec{w} \in W \ \forall k$. $\Rightarrow -\vec{w} \in W$, $0\vec{w} = \vec{0} \in W$

Thus showing 1 & 6 is sufficient to establish W is a subspace of V.

Example: (3) Use the Subspace Test to determine which of the sets are subspaces of M_{nn}.

a. The set of all diagonal $n \times n$ matrices.

b. The set of all $n \times n$ matrices A such that $\det(A) = 0$.

c. The set of all $n \times n$ matrices A such that $\mathrm{tr}(A) = 0$.

d. The set of all symmetric $n \times n$ matrices. — $A = A^T$ — use properties of transpose

Let

a. $\vec{u} = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{bmatrix}$

and $k \in \mathbb{R}$.

i) $\vec{u} + \vec{v} = \begin{bmatrix} a_1+b_1 & & & 0 \\ & a_2+b_2 & & \\ & & \ddots & \\ 0 & & & a_n+b_n \end{bmatrix}$ a diagonal matrix

ii) $k\vec{u} = \begin{bmatrix} ka_1 & & & 0 \\ & ka_2 & & \\ & & \ddots & \\ 0 & & & ka_n \end{bmatrix}$ a diagonal matrix

This is a subspace.

b. Let $A, B \in M_{nn} \ni \det(A) = 0$ and $\det(B) = 0$.

a) Let $A = \begin{bmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$

$\det(A) = 0$, $\det(B) = 0$

But $A+B = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}$ so $\det(A+B) = 1 \neq 0$.

This is not a subspace.

c) Let $A = [a_{ij}]$, $B = [b_{ij}]$ be $\ni$ $tr(A) = tr(B) = 0$

$tr(A) = \sum_{i=1}^{n} a_{ii}$, $tr(B) = \sum_{i=1}^{n} b_{ii}$ $\Rightarrow tr(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii})$

$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = 0 + 0 = 0$

Let $k \in \mathbb{R}$. Then $tr(kA) = \sum_{i=1}^{n}(k a_{ii})$

$= k \sum_{i=1}^{n} a_{ii} = k(0) = 0$

This is a subspace.

Example: (6) Use the Subspace Test to determine which of the sets are subspaces of P_3. $\Rightarrow$ Polynomials of deg ≤ 3

a. All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

b. All polynomials of the form $a_0 + a_1x$ in which a_0 and a_1 are real numbers.

a. Not a subspace. Let $k = \pi$

then $\pi a_0 + \pi a_1 x + \pi a_2 x^2 + \pi a_3 x^3$

does not have rational coefficients.

b. Let $\vec{a} = a_0 + a_1 x$, $\vec{b} = b_0 + b_1 x$

then $\vec{a} + \vec{b} = a_0 + b_0 + a_1 x + b_1 x$

$= (a_0 + b_0) + (a_1 + b_1) x$,

which has the form $c_0 + c_1 x$, $c_0, c_1 \in \mathbb{R}$.

Let $k \in \mathbb{R}$. $k\vec{a} = k a_0 + k a_1 x$, which again has the form $d_0 + d_1 x$, $d_0, d_1 \in \mathbb{R}$.

This is a subspace.

Example: (11) Use the Subspace Test to determine which of the sets are subspaces of M_{22}.

a. All matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$.

b. All matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$.

c. All 2×2 matrices A such that $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Let A & B be 2×2 matrices $\ni A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

and $B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then

$(A+B) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

this is not a subspace

Example: (14) Use the Subspace Test to determine which of the sets are subspaces of R^4.

a. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

b. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where A is as in part (a).

a. Let $\vec{x} \in R^4 \ni A\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $k \in R$.

then $A(k\vec{x}) = kA\vec{x} = k\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

this is not a subspace.

b. Let $\vec{x}, \vec{y} \in R^4 \ni A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $A\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This is closed under vector addition.

Let $k \in R$. Then $A(k\vec{x}) = kA\vec{x} = k\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is closed under scaler multiplication.

This is a subspace. ✓

Using a generic $m \times n$ matrix A, vectors

$\vec{x} \in R^n$, and $\vec{0} \in R^m$, a similar process shows that $\{\vec{x} \in R^n \mid A\vec{x} = \vec{0}\}$ is a subspace.

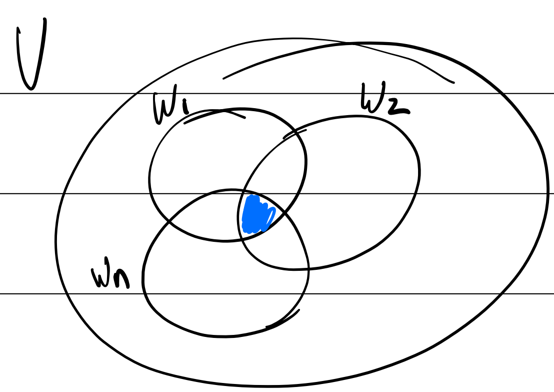
Theorem 4.2.3 The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

Definition: The solution set of a homogeneous system in n unknowns is a subspace of R^n, called the **solution space** of the system.

Definition: Let $T_A : R^n \to R^m$ be multiplication by the coefficient matrix A. The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is called the **kernel** of the transformation.

Theorem 4.2.4 If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Theorem 4.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.



Examples of Subspaces

$\{\mathbf{0}\}$ is a subspace of every vector space V

Any vector space V is a subspace of itself

Subspace of R^2

Lines through the origin $\{(x, y) \mid ax+by=0\}$

Subspaces of R^3

Lines through the origin

Planes through the origin $\{(x, y, z) \mid ax+by+cz=0\}$

The solution space of a homogeneous system in n unknowns is a subspace of R^n

Subspaces of M_{nn}

Symmetric matrices

Triangular matrices

Diagonal matrices

Subspaces of $F(-\infty, \infty)$ (The following is actually a sequence of nested subspaces)

$C(-\infty, \infty)$, the set of functions continuous on R

$C^1(-\infty, \infty)$, the set of functions with continuous first-order derivatives on R

$C^n(-\infty, \infty)$, the set of functions with continuous n^{th}-order derivatives on R

$C^\infty(-\infty, \infty)$, the set of functions with derivatives of all orders on R

P_∞, the set of polynomials

P_n, polynomials of degree $\leq n$